Algorithmic and computer information technologies for modeling the processes of forming classes prime numbers and estimating the constants of Artin`s generalized Hypothesis

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***Abstract*—The dependence between the processes of formation of classes of primes in the generalized hypothesis of Artin is studied on the basis of the theory of randomized algorithms in algebraic and analytic number theories. It is proved that the construction of randomized algorithms for estimating Artin's constants is the basis for the construction of computer models of classes of primes in accordance with the generalized Artin's hypothesis. It has been established that modern methods of analytic number theory do not allow one to obtain estimates for the generalized Artin constants. Methods for calculating Artin's constants for arbitrary sets of natural numbers are developed and the convergence of estimates of constants in probability to limit values is established. The foundations of the number-theoretic analysis of Artin's constants and related classes are created. The dynamics of changes in the laws of distribution of prime numbers depending on the magnitude of the number of prime divisors of the previous number has been studied. A hypothesis is formulated about the limiting properties of the distribution of primitive roots of prime numbers.**

Keywords—generalized Artin classes; Artin constants; class probabilities; stability of estimates for Artin's constants; convergence in probability;

I. INTRODUCTION

The solution of many problems in various areas of applied mathematics depends on the solution of a significant number of problems in pure mathematics that have not yet been solved. Artin's primitive roots conjecture is one of these fundamental mathematical problems. For almost a century, its solution has not been brought to such a level that would allow improving methods for solving the discrete logarithm problem, modern cryptography algorithms, models of algebraic dynamical systems, and developing methods for constructing more advanced pseudo-random number generators.

The solution of Artin's problem is important for studying the relationship between the properties of natural numbers other than ± 1 and k^2 the properties of classes of primes generated by recursive mappings based on Fermat's Little Theorem [1,3]. Another relevant applied problem is the modeling of self-organizing nonlinear dynamic systems, which are commonly called synergetic, taking into account the deep modeling of self-organization phenomena in complex systems consisting of sequences of transitions from one phase state to another using random number generators with a given probability distribution law [3].

Numerical sequences of recursive models of cyclic fixed points of dynamical systems are determined by the properties of prime numbers, with the help of which their behavior is studied under initial conditions $x(0)=1/p$, where p is a prime number. In this case, it is necessary to know the law of distribution of prime numbers. Riemann proposed the zeta function in 1869:

$$\xi(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\in P}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right)$$

where S is a complex variable, P is the set of all prime numbers [1,2]. Regarding this Riemann function, a hypothesis was formed, according to which all non-trivial zeros of this function are on the line $1/2+iy$, where $i=\sqrt{-1}$ and $y\in R$. It follows that all primes lie on this line since y takes values from a set that includes all primes P. Moreover, for any prime number p $\xi(1/2+ip)=0$. In essence, this was the first attempt to find the law of the distribution of prime numbers.

In 1896, Hadamard and Vallée-Poussin independently proved that the equality is true:

$$\pi(x)=\int_2^x\frac{dt}{\ln t}+O\left(x\cdot e^{-c\sqrt{\ln x}}\right) \qquad (1)$$

where $\pi(x)$ is the number of primes $p\le x$, and the first term in the form of a logarithmic smooth function defines the logarithmic distribution of primes in asymptotic form however, this information is not sufficient due to the simple fact that the logarithmic law does not answer the question: is this number n prime or composite. Enumeration of prime numbers with given properties is an even more difficult task [1,2]. In many cases, it becomes necessary to use large arrays of primes with given properties. An expressive example of such a set is the set of all prime numbers for which a given number m is a primitive root, provided that this number is not equal to one and is not a perfect square. The construction of such sets is a very difficult problem in modern number theory.

One of the ways to deepen the logarithmic law of distribution of prime numbers was the formulation in 1927 by the French mathematician Artin of the hypothesis about the primitive roots of prime numbers $p\in P$, and, accordingly, the primitive roots of residue groups $(Z/pZ)^*$ modulo a prime number p [4,7].

Consider the definition of the primitive root of the prime number p. The numbers $a \neq 1$, $a \neq k^2$ is a primitive (primitive) root of the number p if the following relations are satisfied:

$$\begin{cases} a^{p-1} \equiv 1 (\bmod p) \\ a^{\frac{p-1}{n}} \not\equiv 1 (\bmod p), \quad n > 1 \end{cases} \qquad (2)$$

In this case, n is a divisor of $p-1 = \prod_{i=1}^{k} p_i^{\alpha_i}$.

Given the definition of a primitive root, Artin's conjecture is:

$$\pi(x,a) = c(a) \cdot \pi(x) \qquad (3)$$

where $\pi(x,a)$ is the number of primes p less than or equal to x for which $a \neq \pm 1$ and $a \neq k$ are according to (1) their primitive roots, $c(a)$ is Artin's constant. More precisely, this hypothesis should be presented as follows:

$$\begin{cases} \pi(x,a) = c(a,x) \cdot \pi(x), \\ \ln c(a,x) = c(a), \quad x \to \infty \end{cases} \qquad (4)$$

But then $c(a,x) = \frac{\pi(x)}{\pi(x,a,x)}$ converges in probability to $c(a)$, and therefore has a probabilistic interpretation: $c(a)$ is the probability of choosing from the set P a prime number p such that a is its primitive (primitive) root. Note that the first relation in (4) is always satisfied if a and p are coprime numbers according to Fermat's theory [1].

It should be noted that Artin offered his estimates for $c(a)$ at $a = 2$. But as was proved by Hooley [5], these estimates are not correct. He also proved the validity of the relation:

$$\pi(x,2) = \frac{c(2) \cdot x}{\ln x} + O\left(x \frac{\ln\ln(x)}{(\ln(x))^2} \right) \qquad (5)$$

with $c(2) = \prod_{p \in P} \left(1 - \frac{1}{p(p-1)} \right)$, and the score value is $c(2) = 0{,}37395581$. As will be shown later, this estimate is correct only to the accuracy of the first two decimal places.

It should be noted that any number $a > 1$ and coprime to p is the basis for considering the recursive function $f(x) \equiv a \cdot x (\bmod p)$, which leads to a recursive iterative sequence:

$$\begin{aligned} & f(x_0 = 1) = 1 \\ & f(x_{n+1}) = x_{n+1} \equiv a x_n (\bmod p) \end{aligned} \qquad (6)$$

According to Fermat's theorem [1,2], if a is not a primitive root for p, then the process of recursive calculations will continue for such m that $f(x_n = m) \equiv x_{m-1} \cdot a (\mathrm{mod}\, p) = 1$ will be equal i.e.

$$a^m \equiv (\mathrm{mod}\, p) \text{ and } m < p - 1 \qquad (7)$$

From Fermat's theorem and the properties of the group $(Z/pZ)^*$ of residues modulo p [1,2], it follows that in this case a is a generating element of some subgroup of the group $(Z/pZ)^*$. Moreover, m is the order of this subgroup, which is usually denoted by $card_a(p)$, the number of cosets for this subgroup is denoted by $ind_a(p)$. According to the cyclic group theorem $(Z/pZ)^*$, the equality always holds:

$$p - 1 = card_a(p) \cdot ind_a(p) \qquad (8)$$

It follows from the above analysis that Eq. (8) allows us to study Artin's hypothesis from a more general point of view, when any natural number $a > 1$ can be used as a classifier of the set of all prime numbers in terms of $ind_a(p)$, which is the object of further research. As will be established, Artin's primitive roots conjecture is a frequent case of its more general formulation.

II. ANALYSIS OF THE ANALYTIC ASYMPTOTIC METHOD FOR SOLVING THE CLASSICAL ARTIN CONJECTURE

The first attempts to solve Artin's problem are number-theoretic. First of all, it should be noted that they are based on the assumption that the generalized Riemann hypothesis based on the Dedekind zeta function is true. This does not take into account the whole variety of relationships between integers, which can be primitive roots, and the set of all prime numbers for which the selected integer is a primitive root. The importance of this assumption lies in the fact that it is assumed that the Dedekind zeta function takes into account the entire set of variants of the dependence of the formation of classes of all prime numbers for a given number a on its properties, which are described by its decomposition into prime factors. Although the definition of the Dedekind zeta function based on the theory of ideals is based on the Galois theory, this dependence cannot be revealed by analytical methods. In addition, only the primitive roots of prime numbers are taken into account and the generating elements of the subgroups of the cyclic residue group $(Z/pZ)^*$, as well as their properties, are not taken into account. At the same time, information about their properties is important, if only because we get new information about the structure of the group of residues modulo a prime number p.

Probabilistic methods for estimating generalized Artin constants based on computer simulation will be considered as another model for solving this problem and as a generalization of Artin's original hypothesis. The creation of this approach was preceded for a long time by the analysis of the method of its solution by the English mathematician H. Holly, who published in the article

[4], and then re-given for the particular case $a=2$ in the monograph [5]. First of all, let's pay attention to the attitude of well-known specialists in the field of number theory to this solution. The Soviet and American mathematician Yu. I. Manin in his monograph [2] unequivocally asserts that Artin's conjecture has not been proven and is very complicated. The same position is taken by such famous mathematicians as Cohen [6], Rosen [7], Moree [8] and others, but at the same time C. Pomerance in his monograph [1] notes that this problem was solved by Ch. Hooley [4]. Due to this ambiguity of attitude to the work of Ch. Hooley and the importance of its solution from both theoretical and applied points of view, let us dwell on the works of Ch. Hooley [4] in more detail.

Note that to prove the logarithmic distribution of prime numbers (2), the Riemann zeta function (1) was used, but the Riemann hypothesis on the distribution of nontrivial zeros of the zeta function was not used in any way. H. Hooley in [4] and monograph [5] uses the Dedekind zeta function over Galois fields. At the same time, the author assumes that the hypothesis is true:

Hypothesis: The real part β of every complex zero $p=\beta+i\gamma$ Dedekind zeta function is equal to $1/2$ for any Galois field of type G.

This assumption is equivalent in its complexity to the millennium Riemann hypothesis, i.e. Millennium Hypothesis that as provided this assumption is very restrictive. The theorem proved by the author has the form:

Theorem 1. If we assume the validity of the Riemann hypothesis for the zeta functions of Galois fields of type $Q(\sqrt[k_1]{l}, \sqrt[k]{1})$, where k is without square and $k_1 \mid k$, then we have:

a) for any $a \neq 0 \& \pm 1$ other than a perfect square, let $N_a(x)$ be the number of primes $p \leq x$, for each of which a is the primitive root of $\bmod p$, let also a_1 be the without square part of a, let h be the largest integer such that a is h- oh degree of an integer and let:

$$c(h) = \prod_{q|h}\left(1 - \frac{1}{q-1}\right) \cdot \prod_{q \nmid h}\left(1 - \frac{1}{q(q-1)}\right)$$

Then, if $a_1 \not\equiv 1 \pmod 4$, we have:

$$N_a(x) = c(h)\frac{x}{\ln x} + O\left(\frac{x \ln\ln x}{\ln^2 x}\right)$$

at $X \to \infty$, if $a_1 \equiv 1 \pmod 4$, we have:

$$\begin{gathered} N_a(x) = c(h) \cdot (1 - \mu(|a_1|)) \cdot \\ \cdot \prod_{\substack{q|h \\ q|a_1}} \frac{1}{q-2} \cdot \prod_{\substack{q \nmid h \\ q|a_1}} \frac{1}{q^2 - q - 1} \cdot \frac{x}{\ln x} + \\ + O\left(\frac{x \ln\ln x}{\ln^2 x}\right) \end{gathered}$$

at $X \to \infty$.

b) If $a \neq 0, \pm 1$ is not a perfect square, then there are an infinite number of primes p for which a is a primitive root of $\bmod p$.

In the monograph [5] the author repeats the proof only under assumption $a=2$ and the theorem has the following formulation:

Theorem. If we assume that the extended Riemann hypothesis is valid for Dedekind zeta functions over Galois fields and Kummer circular fields of type $Q(\sqrt[k]{2}, \sqrt[k]{1})$, where k is a without square number, then:

a) Let $N_2(x)$ be the number of primes p not exceeding x for which 2 is a primitive root modulo p. Then:

$$N_2(x) = \frac{c \cdot x}{\ln x} + O\left(\frac{x \ln\ln x}{\ln^2 x}\right), \text{ where } c = \prod_q \left(1 - \frac{1}{q(q-1)}\right)$$

b) There are infinitely many primes p for which 2 is a primitive root modulo p.

In proving this theorem, the author states:

“Index theory shows that for $p \neq 2$ and any prime divisor q of $p-1$, the decidability of the comparison is:

$$\upsilon^q \equiv 2 \pmod{p}$$

is equivalent to the divisibility of the index of the number 2 (in any base) by q. Thus, we have a criterion that for each 2 is a primitive root modulo p if and only if $p \neq 2$, and there is no prime divisor q of $p-1$ for which 2 is the residue of the power of q modulo p”.

A comparative analysis of the above two theorems suggests that the author has come to the conclusion that the first theorem is not perfect. It is obvious that for case $a=2$ the expression:

$$c(h) = \prod_{q \mid h}\left(1 - \frac{1}{q-1}\right)\prod_{q \mid h}\left(1 - \frac{1}{q(q-1)}\right)$$

$$c = \prod_q \left(1 - \frac{1}{q(q-1)}\right)$$

do not match. According to Hooli's work, h is the maximum positive integer for which a is the h-th power of an integer modulo p.

Now let's pay attention to several facts that follow from Hooli's theorems and the results of estimating the generalized Artin constants given in the table. Hooli's estimate starting from the third decimal place differs from the exact estimate obtained from the results of computer simulations. It should be noted that the basis of any method for checking whether the number a is a primitive root of a prime number p or a generating element of a subgroup of the residue group $(Z/pZ)^*$ modulo p is a recursive function, which is always determined by the iterative sequence (6). If a natural number a, $a \neq \pm 1$ and a full square, and a prime number p are given, then the length of the recursion (6) in the general case can only be determined as a result of calculations in accordance with this expression. Based on this remark, we can conclude that for different values of the number a for the same prime number p, the lengths of the recursive sequence (6) can differ significantly. The existing information technologies for analyzing the behavior of all pairs (a, p) are not reflected in the generalized Dedekind zeta function. Therefore, an estimate of the Artin constant for a in the case when $a=2$, obtained by analytical methods, can be considered as a value asymptotically close to the exact value, but this estimate of the behavior of this constant on the entire set of pairs (a, p), which is infinite in both a and p, does not reflect the entire possible variety of values.

From the table below, we can conclude that there is a large variety of Artin constants. Hooli's theorems not only do not allow obtaining their estimates, but also do not provide information about their values and the dependence of these values on the properties of primitive roots determined by decomposition into simple factors or their representation in the form $a=4k+1$ or $a=4k+3$ as proposed by Chebyshev [1][2]. Natural numbers other than unity are most interesting to consider as classifiers of the set of primes, according to the index value determined from expression (8). In this case, the perfect squares of natural numbers are the bearer of important information about prime numbers.

III. MODELING THE PROCESSES OF GENERATING DYNAMIC INFORMATION ABOUT THE STRUCTURE OF CLASSES OF PRIME NUMBERS ACCORDING TO A GIVEN BASIS

Now let's return to the logarithmic law of distribution of prime numbers [1,2] in order to pay attention to the fact that the given equality does not provide comprehensive information about the structure of gaps between prime numbers. It is especially important to have information about the distribution of smooth primes [1]. This information is especially important when solving the discrete logarithm problem and applying algorithms for solving it in modern coding theory, modern cryptography. It is known that finding smooth prime numbers of large magnitude is very difficult. It follows from this that it is of considerable interest to search for the laws of distribution of prime numbers not only with respect to their primitive roots, but also with respect to the generating elements of subgroups of the residue group modulo a prime number $(Z/pZ)^*$. Artin's hypothesis does not involve such detailed studies. Problems of this kind were not considered at all.

The second circumstance is that simultaneously with this fact, the dynamics of change in $O\left(x\cdot e^{-\frac{c}{2}\sqrt{\ln x}}\right)$ is investigated. In [7,8], the entropy of function $f(x)=P(x)-Li(x)$ was estimated and was proved that it has a fractal character. These facts are the basis for the formation of proposals on the need to study other models of the distribution of prime numbers.

In addition, it is generally accepted, even at the present time, that it makes sense to study it more fundamentally. The first attempt was made by D. Zagier [8], but not completed. The results obtained by the author confirm the rather complex fractal behavior of this component. Hence it follows that it is necessary to significantly improve the study of the depth of classification of primes, taking into account all models of the formation of classes of primes for any given base $a > 1$. Further more detailed studies of this component confirm that although the logarithmic distribution law is fulfilled, nevertheless, complete information about the dynamic properties of prime numbers and their relationships with their primitive roots still remains poorly studied. In the future, we will consider any values of the base a greater than one.

According to Artin's hypothesis [4–6], the set of such primes has the distribution law $\pi(x,a)$ as an expression:

$$\pi(x,a) = c(a) \cdot \pi(x) \qquad (9)$$

where $\pi(x)$ is the distribution of prime numbers, and $c(a)$ is a constant dependent on a. Until now, despite numerous studies, this hypothesis has not been resolved. However, it is not known if this is true for any a values. If the hypothesis is correct, then the question remains how to estimate the constant $c(a)$ for each concrete a and which properties of the number a influence its value. Answers to these questions are still missing. In works [6][7] a detailed analysis of all the results of research in the field of solving the Artin hypothesis is given.

It should be noted that the proof of Artin's hypothesis is important both from a theoretical point of view in number theory, and from an applied rhenium point, because it’s positive solution is important in cryptography, coding theory, and the theory of dynamical systems. In [6], a generalized Artin hypothesis was formed for any $a > 1$, i.e. and at the same time a may not be a primitive root. According to Artin’s generalized theory, the following equality is true:

$$\pi(x,a,i) = c(a,i) \cdot \pi(x) \qquad (10)$$

where $a > 1$, i is the index of the subgroup of the group $(Z/pZ)^*$ of primes in the classification of prime numbers generated by the numbers a, $c(a,i)$ is a constant. According to the classification built in [6]:

$$P(a,i) = \{ p \in P \mid (p-1)/card_a(p) = i \} \qquad (11)$$

where $card_a(p)$ is the length of the dynamic recursion $x_{n+1} \equiv a x_n \pmod{p}$ at $x_0 = 1$, P is the set of all primes.

It is not difficult to show that for any $a > 1$ the equality:

$$\sum_{i=1}^{\infty} c(a,i) = 1 \qquad (12)$$

This means that primes are evenly distributed in classes $P(a,i)$ for any a. By uniformity is meant that within each class of primes $P(a,i)$ a logarithmic law of the distribution of primes is preserved. The constant $c(a,i)$ determines the measure of puncturing prime numbers, based on the value a. If $i = 1$ then

a is the primitive root of all primes $P(a,1)$. For an arbitrary natural number X, the equality

$$\pi(x,a,i) = d(a,i,x) \cdot \pi(x) \qquad (13)$$

Moreover, if $X \to \infty$, then $d(a,i,x)$ tends to the limit value $d(a,i)$. If we put $i=1$ then $d(a,1)$ will be Artin's constant for primitive roots. In this case $a \neq \pm 1$, and $a \neq k^2$ for none $k \in N$. This is true according to Fermat's theorem [1,2]. Wherein, a is the primitive root of the group of residues $(Z/pZ)^*$ for any $p \in P$ such that $P(a,1) = \{ p \in P | (p-1)/card_a(p) = 1 \}$. It is important to investigate the classes of primes $P(a,i)$ for $i > 1$ since in this case the positive integer a will be the primitive root for the subgroups of the group $(Z/pZ)^*$ with the index defined by the relations:

$$P(a,i) = \{ p | (p-1)/card_a(p) = ind_a(p) \} \quad (14)$$

where $ind_a(p) = i$ is the index of the subgroup of $(Z/pZ)^*$. The classes of primes $P(a,i)$ have not yet been studied and the distribution of primes in these classes is not known. In [1], an assumption was made that $P(a,i)$ at $i > 1$ is proportional to $P(a,1)$ with a factor of $1/i^2$. Since $i > 1$ is considered, in this case it is important to know the distribution of prime numbers for the value $a = k^2$. This is an important generalization of Artin's hypothesis. At the same time, the probability of:

$$\begin{gathered} \mathrm{P}(p \in P(a,i)) = \\ p \in P \| P(a,i) | / | P | = d(a,i) \end{gathered} \quad (15)$$

membership agrees exactly with the provisions of the theory of probability, and therefore, estimating $d(a,i)$ on the basis of successive statistical tests and the law of large numbers is parity [9–12].

The determination of $d(a,i)$ for any a,i using analytical methods is unlikely in the near term. However, the formation and development of experimental mathematics [13–15] opens up another way to solve this problem by using computer simulation of nonlinear dynamic processes for the formation of classes of prime numbers.

The process of modeling the distribution of primes in classes $P(a,1), P(1,2), \ldots, P(a,k), \ldots$ was reduced to choosing a set of consecutive primes from a set of a sufficiently large sample of these classes. The number of primes analyzed at each interval of natural numbers was chosen to be 500,000. This choice was largely due to the fact that it was previously established that reducing this value leads to more significant fluctuations in estimates, although convergence to the limit over the entire set of any intervals, even if they are not placed consistently, has the same character.

The process of statistical testing of $p \in P$ primes for checking their belonging to class $P(a,i)$ was reduced to calculating for the selected number

p the recursive procedure $x_0 = 1$, $x_{n+1} = ax_n \pmod{p}$ until the pairs $ax_i \equiv 1 \pmod{p}$ were reached at some step i. Then $card_a(p) = i$ and according to Fermat's theory and the cyclic group theorem the number $p-1$ is divisible by i and then $ind_a(p) = (p-1)/card_a(p) = i$, and therefore $p \in P(a,i)$ and if $i=1$, then a is the primitive root of the cyclic group $(Z/pZ)^*$, and otherwise it is the primitive root of some subgroup. At $i>1$, we obtain the primitive roots of the subgroups of the $(Z/pZ)^*$ residue group with the index $i>1$.

The study of the distribution law of prime numbers p on their belonging to $P(a,i)$ had the character of consistent statistical tests on the set of natural numbers containing the first 500,000 primes. At the first stage, primes p were chosen from the set $\{p_1, p_2, \ldots, p_{500000}\}$. With this $x = p_{50000}$.

For each $n \in \{2, \ldots, x\}$, we had to solve two problems: check n for simplicity, and if $n = p \in P$, then $p-1$ was decomposed into simple factors, i.e. systematically solved two non-simple problems of checking numbers for simplicity and decomposition into simple factors. An effective algorithm for solving them was created based on probabilistic methods in the theory of elliptic curves.

As a result of analyzing $a \in \{2, \ldots, x\}$, $P(a,1), \ldots, P(a,l)$ sets were obtained for some $l < x$ and absolutely exact values of their powers were calculated, i.e. $|P(a,1)|, \ldots, |P(a,l)|$, and then estimates of:

$$d(a,1,x) = |P(a,1)|/\pi(x), \ldots, d(a,l,x) = |P(a,l,x)|/\pi(x) \qquad (16)$$

while $d(a,1,x) \to d(a,l), \ldots, d(a,l,x) \to d(a,l)$ with $X \to \infty$ were obtained.

At the next stage, work was also carried out for prime numbers from the $\{p_{500001}, \ldots, p_{1000000}\}$ interval and the values of the $d(a,1), \ldots, d(a,l)$ constants were calculated using the same scheme. At the same time l increases. The $\{p_1, \ldots, p_{5000000}\}$ and $\{p_{500001}, \ldots, p_{1000000}\}$ sequences were combined, and the estimates of the generalized Artin constants were again calculated and the process of their refinement was studied on the basis of the theory of large numbers in probability theory. This procedure continued until $x = p = 1794246$ and this is a ten million prime numbers. It was found that $d(a,1), \ldots, d(a,k)$ in probability converges to some values, the exact values of which are irrational and possibly transcendental numbers. In the process of estimating the $d(a,i)$ constants, two important theorems were proved:

Theorem 2. For any $a \in \{2, 3, \ldots, k, \ldots\}$ that is not a square, i.e. $a \neq k^2$ The number of non-empty classes of primes tends to infinity at $X \to \infty$.

Theorem 3. For any $a \in \{2, 3, \ldots, k, \ldots\}$ that is not a square, i.e. $a \neq k^2$ The number of prime numbers in $P(a,i)$ tends to infinity at $X \to \infty$.

These theorems are the basis of the convergence of a sequence of statistical tests to marginal values. Since for any $x \in N$ it is obvious that:

$$\bigcup_{i=1} P(a,i) = \pi(x)$$
$$P(a,i) \cap P(a,j) = \phi$$

at $i \neq j$, it follows from this that:

$$\sum_{i=1}^{k} d(a,i) = 1$$

and this is true for all values of $X \to \infty$. The review [5] provides an estimate of $d(2,1)$, which is identified by $d(2,1)$ in our sense, but $d(2,1)$ differs from the estimate of $d(2,1)$ starting from the fifth decimal place and this is a theoretical error of the survey works.

For different $a \in \{2,3,5,6,7,8,10,11,\ldots\}$, the behavior of the $d(a,i)$ constants is complex group-theoretic and number-theoretic. The study of their dynamic properties is beyond the scope of this work. It should be noted that the results of computer simulation of the processes of distribution of primes are calculated with an accuracy of the eleventh decimal place for estimates of $d(2,1), d(3,1), d(5,1), d(6,1),\ldots$ values. This cannot be asserted for classes by the $i \geq 2$ index. To achieve the same accuracy with $i \geq 2$, it is necessary to significantly increase the number of prime numbers. With an increase in the i class index $P(a,i)$ more than three requirements and the volume of the analyzed primes increases in accordance with the unexplored laws.

Probability-theoretic interpretation of the constant:

$$d(a) = \frac{\pi(x,a)}{\pi(x)} \text{ at } X \to \infty$$

Consider the probability space (Ω, F, P) based on:

$$\Omega = \{\omega_1,\ldots,\omega_n,\ldots\} = \{p_1,\ldots,p_n,\ldots\} = P$$

Obviously at $X \to \infty$ the numbers are $\pi(x) \to \infty$, $\pi(x,a) \to \infty$, but:

$$\pi(x,a) = |P(a,1,x)|$$
$$\pi(x) = |P(x)|$$
$$d(a,1,x) = \frac{|P(a,1,x)|}{|P(x)|}$$

and at $X \to \infty$ it is obvious that:

$$|P(a,1,x)|/|P(x)| \to d(a,1)$$

is where $x \in P$, $P \to \infty$,

$$P(a,i,x) = \{p \mid p \leq x \,\&\, (p-1)/card_a(p) = i\}$$

is at $X\to\infty$ $P(a,i,x)\to P(a,i)$. Thus:

$$d(a)=\lim_{X\to\infty}\pi(x,a)/\pi(x).$$

It follows from Artin's hypothesis that with $d(a,1)$ there is precisely the probability of a random event $P(a,1)$ consisting of a choice of $\Omega=\{p_1,\dots,p_n,\dots\}$ from a prime number p for which a is an original root of the cyclic group $(Z/pZ)^*$. To estimate this probability, the law of large numbers and the method of successive statistical tests were used. The essence of the method is that the first test group was reduced and calculated for $\{p_1,p_2,\dots,p_{500000}\}$ by each $a\in\{2,3,\dots,16\}$ evaluation of the values of $d(a,i,x)$ at $x=p_{500000}$ for all possible values of $i=\{1,2,\dots,k,\dots\}$, that is, $\tilde{c}_1(a,1,x),\dots,\tilde{c}_1(a,k,x),\dots$ was calculated on the next iteration, the same tests were performed for the second iteration on the set $\{p_{500001},\dots,p_{1000000}\}$. $\tilde{c}_1(a,1,x),\dots,\tilde{c}_k(a,1,x),\dots$ Estimates were obtained at the same time $\tilde{c}_1(a,1,x),\dots,\tilde{c}_k(a,k,x),\dots$, provided that the first and second samples were combined and computed values and were determined by $|\tilde{c}(a,i,x)-\tilde{c}(a,1,x)|\le\varepsilon$ for all X. The main focus was on $d(a,1,x)$. As a result of some iterations, it was found that for all a the estimates obtained:

$P(x)=\{p\mid p\le x\}$

$$P(a,i,x)=\{p\mid p\le x\,\&\,(p-1)/card_a(p)=i\}$$

the order of the cyclic group of the subgroup $(Z/pZ)^*$. If $l=p-1$, then a is an original root, and if $l<p-1$ is the original form of the $d(a)$ Artin measure, $d(a,i)$ is a measure of classes by $P(a,i)$ in P. At that $d(a,i)=|P(a,i)|/|P|$ and at the same time:

$$\sum_{i=1}^{\infty}d(a,i)=1\text{ for all } a>1$$

This applies only to classes with indexes $i=1$. For $i\ge 2$ it is necessary to increase the number of statistical tests. This is naturally due to the fact that the classes $P(a,i,x)$ for $i\ge 2$ from numerical theorems contain less than prime numbers. In [1] it is stated that this decrease should be of the order of $1/i^2$ [15], but this is an erroneous assertion. This is clearly seen from “Table 1”. The degree of decline essentially depends on the properties of a and requires a separate study. Case $a\in\{4,9,16\}$ requires separate investigations, because these numbers cannot be primitive roots of that number p, in accordance with the Fermat theorem [3] cannot be generating elements of groups $(Z/pZ)^*$. However, they are generating elements of the subgroups of the group $(Z/pZ)^*$ with even indices. All classes with odd indices are empty sets. “Table 1” shows the constants for $d(a,1)$ for all a except $\{4,9,16\}$. Analysis of the table. The table contains over a thousand columns. The analysis of these data is numerically theoretical and group-specific and goes beyond the scope.

The simulation process of the dynamics of the formation of prime numbers was constructed on the following assumptions. Suppose that an ordered set of prime numbers $P=\{p_1, p_2, \ldots, p_k, \ldots\}$ is given, whose elements are ordered in ascending order. All this set was split into a subset of 500,000 primes. The number of 500,000 is due to the limitations of MS Excel, as a statistical analysis tool, on a number of characteristics of the process of generating prime numbers. Only one restriction is important. We always select 500,000 consecutive primes of the set P. In the current version of Excel, this number can be increased to one million. If you use a powerful computer, you can choose a larger number instead of a million.

The implemented version of the study of dynamic processes for the formation of primes includes the following indicators: the number of a simple number in the p in the ordered set of P, the value of a simple number of p, the value of the recursion length of the numbers $card_a(p)$ at the same value of a for all prime numbers P, the index $ind_a(p)$ of the index of the class:

$$ind_a(p) = (p-1)/card_a(p)$$

the value of the residues modulo any natural module $n>1$, for all classes and any other analytic properties of primes or factors of the decomposition of the number of $p-1$ into simple factors. For each simple multiplier p_i in the:

$$p-1=\prod_{i=1}^{n} p_i^{\alpha_i}$$

Decomposition, one parameter of the dynamic process of generating primes is presented, with separate indicators that can be analyzed for any other indicators, the values for them are deducted by the modulus of the natural number $n>1$. The only exception is $ind_a(p)$. The number of controlled indicators analyzed in the Excel environment can be expanded.

According to the idea of experimental mathematics on the first iteration, we proceed from hypothetically known data. But it is also the basis for obtaining experimental information on the basis of which the analytical methods of the theory of numbers yield an expanded representation of the hypothesis in the form H_i. It is possible that at the same time the hypothesis can be corrected or even rejected as not true. From the point of view of information technology in mathematics, the hypothesis H_i is used to develop from the point of view of deepening the experimental mathematics of the model of in-depth studies at the level I_1.

The iterations process is continued until an analytically based solution of the generated hypothesis is obtained. Since the Artin generalized hypothesis is considered in the paper, we present the results of the estimation of the constant $d(a,i)$ for the case $a=4$ and $i=2$. The number $a=4$ is a perfect square, and therefore it cannot be a primitive root. In terms of Artin's generalized hypothesis, this is as interesting and important as in the case when a is an original root.

TABLE I. THE QUANTITY OF PRIME NUMBERS INTO INTERVALS FOR A=2,3,5,8,12

Interval / a	**2**	**3**	**5**	**8**	**12**
(0.0 – 0.5) 10^6	187111	187011	196980	112331	187013
(0.5 – 1.0)·10^6	186912	186948	196836	112075	187057
(1.0 – 1.5)·10^6	186953	186960	197030	112175	187040
(1.5 – 2.0)·10^6	186846	186856	196894	112201	186958
(2.0 – 2.5)·10^6	187410	186896	196720	112345	186792
(2.5 – 3.0)·10^6	186711	186777	196957	112042	186767
(3.0 – 3.5)·10^6	187096	186926	197025	112335	187157
(3.5 – 4.0)·10^6	186975	187176	196942	112283	186984
(4.0 – 4.5)·10^6	187197	187148	196543	112296	187317
(4.5 – 5.0)·10^6	186713	186796	196689	121919	186721
(5.0 – 5.5)·10^6	186828	187013	197050	112093	187005
(5.5 – 6.0)·10^6	187197	186771	196790	112362	186936
(6.0 – 6.5)·10^6	186881	187116	196851	112226	187056
(6.5 – 7.0)·10^6	187065	187214	196478	112093	187122
(7.0 – 7.5)·10^6	187039	186718	196957	112236	187050
(7.5 – 8.0)·10^6	187045	186756	196764	112128	187161
(8.0 – 8.5)·10^6	187299	186805	196840	112187	186594
(8.5 – 9.0)·10^6	186663	187050	196583	111967	187144
(9.0 – 9.5)·10^6	186874	187156	196795	112133	186976
(9.5 – 10.0)·10^6	187034	187072	197083	111993	186947
10^8 – 10^8+0.5·10^6	186932	187417	196902	112048	186555
10^9 – 10^9 + 0.5 10^6	187123	186897	197070	112386	187073
10^{12} – 10^{12} + 0.5·10^6	187127	186994	196871	112303	186786

Based on the data presented in [6], we obtained estimates for $d(a,i)$ and $a=2,3,\ldots 9,10$ and $i=1,2,\ldots 9,\ldots$. It is shown that their values are stable for class $P(4,2)$ i.e. class with $ind_4(p)=2$ to within a fourth decimal place.

The estimates for the $d(a,i)$ constants given in “Table 1” have the unique $i=1$ property, which is that for $a\in\{2,3,5,6,7,10,11,12,13,14,15\}$ they coincide with the accuracy of the third decimal place.

The data in “Table 1” allow us to make an important conclusion that there are many primitive roots for which the generalized Artin`s constant $d(a,1)$ is equal to the same value $0.3739\ldots$ The generalized Artin hypothesis for all classes $P(a,1),\ldots,P(a,i),\ldots$ will require additional studies based on probabilistic computer simulation on the set of prime numbers of data beyond the limits of the first hundred million.

The results of experimental mathematics in “Table 1” of the first iteration confirm that Artin's hypothesis is correct. The estimates of the constants are obtained with the accuracy of the third decimal place. For $a\in\{2,3,5,6,7,8,10\}$ the:

$$\sum_{i=1}^{\infty} d(a,i)=1$$

and for $a\in\{4,9\}$ all $d(a,2i+1)=0$ and:

$$\sum_{i=1}^{\infty} d(a,2i)=1$$

This is due to the fact that for all $a=k^2$ this is true because they are primitive roots of $(Z/pZ)^*$ groups, but primitive roots of their subgroups with even indices [3].

The results obtained are the basis for constructing an analytical proof of Artin’s hypothesis and its general form without using the Dedekind zeta function, which will be presented in the next works of the authors.

The $d(a,1)$ ratings given in the table for the set of primitive roots $\{2,3,\dots,16\}$ are obtained for the first time based on the results of computer simulation. The literature is known estimation $d(2,1)$, which, starting from the fourth decimal place, is estimated analytically incorrect, due to the fact that the formula:

$$d(2,1)=\prod_{p\in P}\left(1-\frac{1}{p\cdot(p-1)}\right)$$

is not true, because it includes all primes and among them those primes for which $a=2$ is not a primitive root [5]. An important result is the creation of a computer model of the process of forming classes $P(a,1),\dots,P(a,i),\dots$. For any values of $a>1$, the interactions between the classes “Table 2” and “Table 3” are investigated (as a continuation). The first estimates were $d(a,i)$ for $i\geq 2$, and it was established that the statement that $d(a,i)$ is proportional to $1/i^2$ is absolutely false [1]. Obtaining the results is the basis for further deepening research on the Artin's hypothesis using analytical methods.

The numbers $a=3$ and $a=12$ are included in “Table 1” for the reason that $P(3,1)$ and $P(12,1)$ contain 0.8 common primes, while for any other pairs of sets $P(a_l,1)$ and $P(a_s,1)$, the total fraction of common primes is 0.4 for $l\neq s$ from the considered set of values. These two facts are obtained on the basis of the analysis of the information presented in “Table 3”. This result was also based on the methods of modern number theory and probability theory.

TABLE II. TABLE OF GENERALIZED ARTIN CONSTANTS WITH INDEX I VALUES LESS THAN OR EQUAL TO 10

a	***P(a,1)***	***P(a,2)***	***P(a,3)***	***P(a,4)***	***P(a,5)***	***P(a,6)***	***P(a,7)***	***P(a,8)***	***P(a,9)***	***P(a,10)***
2	0,3740	0,2805	0,0664	0,0467	0,0189	0,0498	0,0089	0,0351	0,0074	0,0141
3	0,3739	0,2992	0,0666	0,0561	0,0190	0,0332	0,0089	0,0140	0,0074	0,0151

4	0	0,5609	0	0,0935	0	0,0997	0	0,0701	0	0,0283
5	0,3937	0,2657	0,0700	0,0664	0	0,0473	0,0094	0,0166	0,0078	0,0284
6	0,3741	0,2805	0,0665	0,0748	0,0189	0,0498	0,0089	0,0140	0,0074	0,0142
7	0,3741	0,2827	0,0664	0,0684	0,0188	0,0503	0,0089	0,0170	0,0074	0,0143
8	0,2243	0,1683	0,1995	0,0281	0,0114	0,1496	0,0054	0,0211	0,0222	0,0085
9	0	0,5983	0	0,1122	0	0,0666	0	0,0281	0	0,0303
10	0,3741	0,2804	0,0665	0,0713	0,0189	0,0499	0,0089	0,0166	0,0074	0,0142
11	0,3741	0,2813	0,0664	0,0695	0,0189	0,0500	0,0089	0,0173	0,0074	0,0142
12	0,3740	0,2991	0,0665	0,0561	0,0189	0,0333	0,0090	0,0140	0,0074	0,0152
13	0,3764	0,2787	0,0670	0,0697	0,0191	0,0495	0,0090	0,0174	0,0074	0,0141
14	0,3739	0,2806	0,0665	0,0707	0,0189	0,0498	0,0089	0,0171	0,0074	0,0141
15	0,3739	0,2796	0,0665	0,0708	0,0189	0,0508	0,0089	0,0177	0,0074	0,0151
16	0	0,3740	0	0,1869	0	0,0664	0	0,1403	0	0,0189
17	0,3754	0,2794	0,0667	0,0698	0,0190	0,0497	0,0090	0,0175	0,0075	0,0141
18	0,3740	0,2805	0,0664	0,0467	0,0189	0,0498	0,0089	0,0350	0,0074	0,0142
19	0,3739	0,2808	0,0665	0,0700	0,0189	0,0499	0,0089	0,0175	0,0074	0,0142
20	0,3936	0,2657	0,0700	0,0664	0	0,0472	0,0094	0,0166	0,0078	0,0284
21	0,3722	0,2819	0,0681	0,0705	0,0188	0,0486	0,0107	0,0176	0,0076	0,0142
22	0,3740	0,2805	0,0665	0,0704	0,0189	0,0499	0,0089	0,0174	0,0074	0,0141
23	0,3741	0,2808	0,0664	0,0699	0,0189	0,0499	0,0089	0,0175	0,0074	0,0141
24	0,3740	0,2805	0,0665	0,0748	0,0189	0,0498	0,0089	0,0140	0,0074	0,0142
25	0	0,5708	0	0,1328	0	0,1015	0	0,0333	0	0,0190
26	0,3741	0,2805	0,0664	0,0702	0,0189	0,0499	0,0090	0,0174	0,0074	0,0142
27	0,2244	0,2244	0,1994	0	0,0113	0,0997	0,0054	0	0,0222	0,0114
28	0,3740	0,2828	0,0665	0,0684	0,0188	0,0503	0,0090	0,0171	0,0074	0,0143
29	0,3745	0,2801	0,0666	0,0700	0,0189	0,0498	0,0089	0,0176	0,0074	0,0141
30	0,3740	0,2805	0,0665	0,0699	0,0189	0,0499	0,0089	0,0178	0,0074	0,0142
31	0,374	0,280	0,066	0,070	0,018	0,049	0,008	0,017	0,007	0,014

	1	6	5	1	8	9	9	5	4	2
32	0,2953	0,2214	0,0524	0,0369	0,0945	0,0394	0,0070	0,0277	0,0058	0,0709
53	0,3740	0,2804	0,0665	0,0701	0,0190	0,0498	0,0090	0,0175	0,0074	0,0142

In conclusion, by returning attention to “Table 1” of another theory of vision. The essence of a fundamentally new fact is that wherever 500,000 primes $p\in P$ are selected for any $a>$, the number of primes in classes ranges from no more than 500, which is no more than a thousandth of them. This means that on any set of consecutive primes we obtain an estimate of the Artin’s constants up to the fifth decimal place. Statistical summation of values over the entire set of the first ten million primes made it possible to obtain estimates of the constants $d(a,1)$ to the eighth decimal place.

It follows that the methods of computer simulation of the processes of forming classes of primes $P(a,1),P(a,2),\dots,P(a,i),\dots$ and estimation of constants $d(a,1),d(a,2),\dots,d(a,i),\dots$ are the basis for the development of information technologies in modern both pure and applied mathematicians.

“Table 2” shows the values of the estimates of the generalized Artin’s constants for the marked set of values of a, which were studied as classifiers of the set of all primes.

“Table 2” shows a fragment of the entire huge number of obtained estimates of the Artin’s constants. The first column contains estimates of Artin’s constants in its original form. They belong to primitive roots. The numbers $4,9,16$ are not primitive roots, since according to Fermat’s theorem they, like the squares of other numbers, cannot be primitive roots. However, they can be classifiers of primes as roots of subgroups of residues modulo primes. An interesting fact is that they can be used to build pseudo-random number generators. In addition, the diskette logarithm problem can be considered on their basis.

IV. DYNAMIC PROPERTIES OF THE FORMATION OF CLASSES OF PRIMES IN THE GENERALIZED ARTIN CONJECTURE

In accordance with the developed mathematical model for the formation of classes of primes by base $a>1$ and the calculated values of the generalized constants $d(a,i)$ for $i\geq 1$, as a result of computer simulation, it was found that the generalized hypothesis is correct. “Table 1” shows the values of Artin's constants, relationships between classes, the dynamics of class formation and its properties on the set of all primes P.

In reality, the modeling of the processes of formation of classes $P(a,i)$ was carried out for the set $a\in\{2,3,\dots,17,29,53\}$. The numbers $a\in\{4,9,16\}$ as squares of numbers, according to Fermat's theorem [2], cannot be primitive roots of $p\in P$, and, accordingly, the residue groups of $(\mathbb{Z}/p\mathbb{Z})^*$ modulo p. Particular attention was paid to the numbers $\{5,13,17,29,53\}$ due to the fact that they belong to the class of numbers of the Chebyshev type [1,2], that is, they have representations $p=4k+1$, while $p\in P$, and the number n is a natural number. According to Chebyshev's assumption, the behavior of these numbers in the classes of residues modulo a prime number must differ from other prime numbers.

To solve the problem of modeling classes of primes on a given basis and estimating the generalized Artin constants $C(a,i)$, an Excel-based software package was created that allows you to extend the modeling process to any natural numbers $a>1$, and any set of consecutive primes whose power is a multiple of 500,000. This is the number of primes numbers was chosen because it is statistically representative and provides an accurate representation of the dynamic processes that form the $P(a,i)$ classes. “Table 1” shows the results of the simulation process for $a\in\{2,3,5,8,12\}$ values. This set includes $a=2$ for the reason that it can be seen that the estimate [5,6] differs from the exact value. The difference starts at the third decimal place. This fact is important due to the fact that expression (4), although from an asymptotic point of view, is close to the exact value of $C(2)$, nevertheless, it does not take into account all the features of the formation of classes $P(a,1)$ for $a=2$. The number $a=5$ is interesting because $a=5=4\cdot1+1$ is the smallest Chebyshev number that is most sensitive to the established fact that all classes $P(5,10k+5)$ for $k\geq0$ are empty. This is true for all Chebyshev numbers. The proof of this fact is number-theoretic, and therefore, is not given.

The numbers $8,27,32$ are interesting because the dynamical properties of the classes $P(8,i)$ are radically different from those of the other classes studied. In particular, it is established that if $a=8$ is a primitive root of $p\in P$ then $a=2$ is also a primitive root of the same prime number. Conversely, if $a=2$ is a primitive root of $p\in P$, then $a=8$ will be either a similarly primitive root of p or $p\in P(8,3)$. This is brand new information about Artin's generalized constants. The developed approach made it possible to obtain fundamentally new results in modern number theory, and, as a result, in modern cryptography.

“Table 2” shows estimates of the generalized Artin constants. The first column contains a list of $a\in\{2,3,\ldots,3253\}$ values, as an indication that this is reaching the limit of prime numbers of type $p=4k+1$. The second column and subsequent columns correspond to Artin's constants in his interpretation for the $ind(a,p)\in\{1,2,\ldots,10\}$ index values. The second column gives estimates of Artin's constants for values of a as a primitive prime root of ten million consecutive primes. Starting from the third column, estimates of the generalized Artin constants are found for the reduced index values equal to the number 2 and up to 10. A preliminary analysis for the numbers $a=5$ and $a=20$ shows that the dynamics of all generalized Artin constants is almost identical. The existence of windows with zeros for these values is a simple rationale for increasing the values of Artin's constants, which was not justified at the time of the analysis of this problem by both Artin and Bilhartz. Moreover, at $a=8$ and $a=27$ the Artin`s constants are the same, but the behavior of $ind(a,p)>1$ these values is very different.

Let's take a look at $a=32$. The generalized Artin constants take on completely different values. This implies that the theory of the Artin hypothesis in a generalized form requires a significant deepening and development for all values of $a > 2$.

In closing, looking back at “Table 1” from a different perspective. The essence of a fundamentally new fact is that no matter where 500,000 prime numbers $p \in P$ are sequentially allocated, for any a more, the number of prime numbers in classes varies within no more than 500, which is no more than a thousandth of them. This means that on any set of consecutive primes we obtain an estimate of Artin's constants up to the fifth decimal place. Summation by statistical methods of the values over the entire set of the first ten million primes made it possible to obtain estimates of the constants $\delta(a,1)$ with an accuracy of eight decimal places.

Hence it follows that the methods of computer simulation of the processes of forming classes of prime numbers $P(a,1), P(a,2), \ldots, P(a,i), \ldots$ and estimating constants $\delta(a,1), \delta(a,2), \ldots, \delta(a,i), \ldots$ are the basis for the development of information technologies in modern both pure and applied mathematics.

An interesting result is that the constants:

$$\delta(2,1) = \delta(3,1) = \delta(6,1) = \delta(7,1) =$$
$$\delta(10,1) = \ldots = \delta(15,1) = \delta(17,1)\ldots$$

are equal to one thousandth, although $\delta(8,1)$ and $\delta(5,1)$ are radically different from them. On the basis of modern number theory and the theory of random processes, the validity of such results is proved. The proofs of these statements can be built only on the basis of data obtained as a result of computer simulation.

Now consider for all prime numbers of the given sets the dynamics of the change in the number of different prime factors into which the numbers $p-1$ are decomposed. Work in the Excel environment when creating a modeling environment in which all information is available regarding the set of all characteristics of the processes of formation of generalized Artin constants. “Table 3” shows the dynamics of changes in the value of $\omega(p-1)$. In the first three columns for $\omega(p-1)$, the numbers of primes decrease, but starting from $\omega(p-1) > 4$, these numbers increase. You should pay attention to the simple fact that the more $\omega(p-1)$ the greater the rate of increase. But this means that the number of primitive roots in sets of consecutive primes will decrease. If this assumption is correct, then Artin's hypothesis will have to be substantially corrected.

TABLE III. DISTRIBUTION OF VALUES $\omega(p-1)$

Interval / ω(p-1)	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$(0.0 – 0.5) \cdot 10^6$	50381	167691	188701	84851	11437	299	-	-
$(0.5 – 1.0) \cdot 10^6$	44852	153574	188257	94856	17657	772	2	-

(1.0 – 1.5) $\cdot 10^6$	43143	149621	186880	98901	20324	1125	6	-
(1.5 – 2.0) $\cdot 10^6$	42049	147051	186006	101436	22076	1374	8	-
(2.0 – 2.5) $\cdot 10^6$	41330	145333	185460	102972	23374	1517	14	-
(2.5 – 3.0) $\cdot 10^6$	40780	143601	185245	104499	24123	1736	16	-
(3.0 – 3.5) $\cdot 10^6$	40326	142686	184372	105823	24927	1926	20	-
(3.5 – 4.0) $\cdot 10^6$	40165	141304	184177	106508	25782	2028	36	-
(4.0 – 4.5) $\cdot 10^6$	39694	140883	183811	107071	26326	2187	28	-
(4.5 – 5.0) $\cdot 10^6$	39231	139896	184192	107351	26879	2276	35	-
(5.0 – 5.5) $\cdot 10^6$	39043	139688	183484	108162	27238	2349	36	-
(5.5 – 6.0) $\cdot 10^6$	38960	138909	183247	108435	27925	2487	37	-
(6.0 – 6.5) $\cdot 10^6$	38626	138412	182768	109237	28343	2563	51	-
(6.5 – 7.0) $\cdot 10^6$	38513	137593	182791	109804	28687	2560	52	-
(7.0 – 7.5) $\cdot 10^6$	38544	136788	182836	109955	29130	2705	42	-
(7.5 – 8.0) $\cdot 10^6$	38011	136859	182333	110612	29371	2759	55	-
(8.0 – 8.5) $\cdot 10^6$	37991	136071	182209	110803	29475	2853	62	-
(8.5 – 9.0) $\cdot 10^6$	37814	136240	182186	110993	29823	2887	57	-
(9.0 – 9.5) $\cdot 10^6$	37988	135473	182142	111187	30192	2952	66	-
(9.5 – 10.0) $\cdot 10^6$	37330	135688	181949	111720	30252	3003	58	-
10^8 – 10^8+0.5 $\cdot 10^6$	33052	122584	175970	121119	40710	6235	328	2
10^9 – 10^9 + 0.5 $\cdot 10^6$	29453	112182	170284	127551	49907	9791	810	22
10^{12} – 10^{12} + 0.5 $\cdot 10^6$	22339	91324	153222	136364	70176	21575	3643	348

$\omega(p-1)$ is the number of different prime divisors of the number $p-1$.

ACKNOWLEDGMENT

Based on the analysis of the processes of formation of classes of prime numbers, fundamentally new information technologies for solving complex mathematical problems using the methods of modern experimental mathematics were created for any bases. The correctness of the developed approach and computational efficiency are proved. A generalized theory of the Artin hypothesis has been developed, in which its classical version is a very special case. Estimates of Artin's constants for bases greater than the number two are obtained, and the statistical consistency of the obtained estimates is proved. A detailed analysis of the classes of prime numbers has been carried out and the foundations of effective methods for the structural analysis of classes have been created. It is proved that a new method for modeling the dynamics of the formation of classes of prime numbers and describing their properties creates the basis for constructing more advanced models of pseudo prime number generators, the development of new methods for protecting information in modern cryptography, and opens up new opportunities for building models of nonlinear dynamic systems.

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